

THE CENTER OF THE CATEGORY OF BIMODULES AND DESCENT DATA FOR NON-COMMUTATIVE RINGS

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ABSTRACT. Let A be an algebra over a commutative ring k . We compute the center of the category of A -bimodules. There are six isomorphic descriptions: the center equals the weak center, and can be described as categories of noncommutative descent data, comodules over the Sweedler canonical A -coring, Yetter-Drinfeld type modules or modules with a flat connection from noncommutative differential geometry. All six isomorphic categories are braided monoidal categories: in particular, the category of comodules over the Sweedler canonical A -coring $A \otimes A$ is braided monoidal. We provide several applications: for instance, if A is finitely generated projective over k then the category of left $\text{End}_k(A)$ -modules is braided monoidal and we give an explicit description of the braiding in terms of the finite dual basis of A . As another application, new families of solutions for the quantum Yang-Baxter equation are constructed: they are canonical maps Ω associated to any right comodule over the Sweedler canonical coring $A \otimes A$ and satisfy the condition $\Omega^3 = \Omega$. Explicit examples are provided.

INTRODUCTION

A monoidal category can be viewed as a categorical version of a monoid. The appropriate generalization of the center of a monoid is given by the centre construction, which was introduced independently by Drinfeld (unpublished), Joyal and Street [13] and Majid [17]. A key result in the classical theory is the following: the center of the category of representations of a Hopf algebra H is isomorphic to the category of Yetter-Drinfeld modules over H [14]. Moreover, if the Hopf algebra H is finite dimensional, then the category of Yetter-Drinfeld modules is isomorphic to the category of representations over the Drinfeld double $D(H)$. Since the center is a braided monoidal category, it follows that the Drinfeld double is a quasitriangular Hopf algebra.

Let A be an algebra over a commutative ring k . In this note, we study the center of the category ${}_A\mathcal{M}_A$ of A -bimodules, and relate it to some classical concepts. We introduce $A \otimes A^{\text{op}}$ -Yetter-Drinfeld modules (Definition 2.1), and show that the weak center of ${}_A\mathcal{M}_A$ is isomorphic to the category of $A \otimes A^{\text{op}}$ -Yetter-Drinfeld modules (Proposition 2.3). We give other descriptions: the weak center is equal to the center (Proposition 2.6) and is isomorphic to the category $\mathcal{M}^{A \otimes A}$ of comodules over the Sweedler canonical coring $A \otimes A$ (Proposition 2.2). Moreover it was proved in [6, Theorem 5] that the category

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$\mathcal{M}^{A \otimes A}$ is isomorphic to the category of right A -modules with a flat connection as defined in noncommutative differential geometry. Thus, the category of right A -modules with a flat connection is also equal to the center. We introduce a category of descent data $\underline{\text{Desc}}(A/k)$, generalizing the descent data introduced in [15] from A commutative to A non-commutative, and this category is also isomorphic to the center. The first main result of this paper is summarized in Theorem 2.10 which provide six isomorphic descriptions for the center of the category of A -bimodules. All six isomorphic categories are braided monoidal categories. In particular, the category of comodules over the Sweedler canonical A -coring $A \otimes A$ is a braided monoidal category and hence one can perform most of the constructions that are performed for differentiable manifolds. For instance, connections in bimodules try to mimic linear connections in geometry and are useful in capturing Riemannian aspects (see [8], [7] for more details). If A is faithfully flat as a k -module, all these categories are equivalent to the category of k -modules, by classical descent theory. In the case where A is finitely generated and projective, the category $\mathcal{M}^{A \otimes A}$ is isomorphic to the category of left modules over $\text{End}_k(A)$, in fact, one may view $\text{End}_k(A)$ as the Drinfeld double of the enveloping algebra $A^e = A \otimes A^{\text{op}}$. Thus, the category of left $\text{End}_k(A)$ -modules is braided monoidal, and we give an explicit description of the monoidal structure and the braiding in terms of the finite dual basis of A . If we apply this to the case where $A = k^n$, then we find that the category of left modules over the matrix ring $M_n(k)$ is braided monoidal. We give an explicit description of the tensor product and the braiding.

The second major application of the above results is the fact that they lead to constructing new and interesting family of solutions for the quantum Yang-Baxter equation. If V is a right comodule over the Sweedler canonical coring $A \otimes A$, then the canonical map $\Omega : V \otimes V \rightarrow V \otimes V$, $\Omega(v \otimes w) = w_{[0]} \otimes v_{[0]} w_{[1]} v_{[1]}$ is a solution of the quantum Yang-Baxter equation and $\Omega^3 = \Omega$ in the endomorphism algebra $\text{End}(V \otimes V)$ (Theorem 4.1). Several examples are provided.

1. PRELIMINARY RESULTS

1.1. Braided monoidal categories and the center construction. A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, I, a, l, r)$ consists of a category \mathcal{C} , a functor $\otimes : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, called the tensor product, an object $I \in \mathcal{C}$ called the unit object, and natural isomorphisms $a : \otimes \circ (\otimes \times \mathcal{C}) \rightarrow \otimes \circ (\mathcal{C} \times \otimes)$ (the associativity constraint), $l : \otimes \circ (I \times \mathcal{C}) \rightarrow \mathcal{C}$ (the left unit constraint) and $r : \otimes \circ (\mathcal{C} \times I) \rightarrow \mathcal{C}$ (the right unit constraint). a , l and r have to satisfy certain coherence conditions, we refer to [14, XI.2] for a detailed discussion. \mathcal{C} is called strict if a , l and r are the identities on \mathcal{C} . McLane's coherence Theorem asserts that every monoidal category is monoidal equivalent to a strict one, see [14, XI.5]. The categories that we will consider are - technically spoken - not strict, but they can be threatened as if they were strict.

Let $\tau : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C} \times \mathcal{C}$ be the flip functor. A prebraiding on \mathcal{C} is a natural transformation $c : \otimes \rightarrow \otimes \circ \tau$ satisfying the following equations, for all $U, V, W \in \mathcal{C}$:

$$c_{U, V \otimes W} = (V \otimes c_{U, W}) \circ (c_{U, V} \otimes W) ; \quad c_{U \otimes V, W} = (c_{U, W} \otimes V) \circ (U \otimes c_{V, W}).$$

c is called a braiding if it is a natural isomorphism. c is called a symmetry if $c_{U,V}^{-1} = c_{V,U}$, for all $U, V \in \mathcal{C}$. We refer to [14, XIII.1], [12] for more details.

There is a natural way to associate a (pre)braided monoidal category to a monoidal category. The weak right center $\mathcal{W}_r(\mathcal{C})$ of a monoidal category \mathcal{C} is the category whose objects are couples of the form $(V, c_{-,V})$, with $V \in \mathcal{C}$ and $c_{-,V} : - \otimes V \rightarrow V \otimes -$ a natural transformation such that $c_{-,I}$ is the natural isomorphism and $c_{X \otimes Y, V} = (c_{X,V} \otimes Y) \circ (X \otimes c_{Y,V})$, for all $X, Y \in \mathcal{C}$. The morphisms are defined in the obvious way. $\mathcal{W}_r(\mathcal{C})$ is a prebraided monoidal category; the unit is (I, id) , and the tensor product is

$$(V, c_{-,V}) \otimes (V', c_{-,V'}) = (V \otimes V', c_{-,V \otimes V'})$$

where

$$c_{X, V \otimes V'} = (V \otimes c_{X, V'}) \circ (c_{X, V} \otimes V')$$

for all $X \in \mathcal{C}$. The prebraiding is given by

$$c_{V, V'} : (V, c_{-,V}) \otimes (V', c_{-,V'}) \rightarrow (V', c_{-,V'}) \otimes (V, c_{-,V})$$

for all $V, V' \in \mathcal{C}$. The right center $\mathcal{Z}_r(\mathcal{C})$ is the full subcategory of $\mathcal{W}_r(\mathcal{C})$ consisting of objects $(V, c_{-,V})$ with $c_{-,V}$ a natural isomorphism; $\mathcal{Z}_r(\mathcal{C})$ is a braided monoidal category. For more detail, we refer to [14, XIII.4].

1.2. Descent data. Let A be a commutative k -algebra. \otimes will always mean \otimes_k , and $A^{(n)}$ will be a shorter notation for the n -fold tensor product $A \otimes \cdots \otimes A$. If V and W are right A -modules, then $V \otimes W$ is a right $A^{(2)}$ -module. Consider a map $g : A \otimes V \rightarrow V \otimes A$ in $\mathcal{M}_{A^{(2)}}$. For $a \in A$ and $v \in V$, we write - temporarily - $g(a \otimes v) = \sum_i v_i \otimes a_i$. Then we have the following three maps in $\mathcal{M}_{A^{(3)}}$

$$(1) \quad \begin{aligned} g_1 : A \otimes A \otimes V &\rightarrow A \otimes V \otimes A & ; & \quad g_1(b \otimes a \otimes v) = \sum_i b \otimes v_i \otimes a_i; \\ g_2 : A \otimes A \otimes V &\rightarrow V \otimes A \otimes A & ; & \quad g_2(a \otimes b \otimes v) = \sum_i v_i \otimes b \otimes a_i; \\ g_3 : A \otimes V \otimes A &\rightarrow V \otimes A \otimes A & ; & \quad g_3(a \otimes v \otimes b) = \sum_i v_i \otimes a_i \otimes b. \end{aligned}$$

Let $\psi : V \otimes A \rightarrow V$ be the right A -action on V .

Proposition 1.1. [15, Prop. II.3.1] *Assume that $g_2 = g_3 \circ g_1$. Then g is an isomorphism if and only if $\psi(g(1 \otimes v)) = v$, for all $v \in V$.*

In this situation, (V, g) is called a descent datum. A morphism between two descent data (V, g) and (V', g') is a right A -linear map $f : V \rightarrow V'$ such that $(f \otimes A) \circ g = g' \circ (A \otimes f)$. The category of descent data is denoted by $\underline{\text{Desc}}(A/k)$. We have a pair of adjoint functors (F, G) between \mathcal{M}_k and $\underline{\text{Desc}}(A/k)$. For $N \in \mathcal{M}_k$, $F(N) = (N \otimes A, g)$, with $g(a \otimes n \otimes b) = n \otimes a \otimes b$. $G(V, g) = \{v \in V \mid v \otimes 1 = g(1 \otimes v)\}$. The unit and counit of the adjunction are as follows:

$$\eta_N : N \rightarrow (GF)(N), \quad \eta_N(n) = n \otimes 1;$$

$$\varepsilon_{(V, g)} : (FG)(V, g) = G(V, g) \otimes A \rightarrow (V, g), \quad \varepsilon_{(V, g)}(v \otimes a) = va.$$

The Faithfully Flat Descent Theorem can now be stated as follows: if A is faithfully flat over k , then (F, G) is an inverse pair of equivalences. This is essentially [15, Théorème 3.3], formulated in a categorical language. In [15], a series of applications of descent theory are given, and there exist many more in the literature. Also observe that the

descent theory presented in [15] is basically the affine version of Grothendieck's descent theory [11].

1.3. Noncommutative descent theory and comodules over corings. Descent theory can be extended to the case where A are noncommutative. This was done by Cipolla in [10]. After the revival of the theory of corings initiated in [5], it was observed that the results in [10] can be nicely reformulated in terms of corings. Recall that an A -coring \mathcal{C} is a coalgebra in the monoidal category of A -bimodules. A right \mathcal{C} -comodule is a right A -module M together with a right A -linear map $\rho : M \rightarrow M \otimes_A \mathcal{C}$ satisfying appropriate coassociativity and counit conditions. For detail on corings and comodules, we refer to [5, 8]. An important example of an A -coring is Sweedler's canonical coring $\mathcal{C} = A \otimes A$. Identifying $(A \otimes A) \otimes_A (A \otimes A) \cong A^{(3)}$, we view the comultiplication as a map $\Delta : A^{(2)} \rightarrow A^{(3)}$. It is given by the formula $\Delta(a \otimes b) = a \otimes 1 \otimes b$. The counit ε is given by $\varepsilon(a \otimes b) = ab$. For a right A -module M , we can identify $M \otimes_A (A \otimes A) \cong M \otimes A$. A right $A \otimes A$ -comodule is then a right A -module V together with a right k -linear map $\rho : V \rightarrow V \otimes A$, notation $\rho(v) = v_{[0]} \otimes v_{[1]}$ satisfying the relations

$$\begin{aligned} (2) \quad & v_{[0]}v_{[1]} = v; \\ (3) \quad & \rho(v_{[0]}) \otimes v_{[1]} = v_{[0]} \otimes 1 \otimes v_{[1]}; \\ (4) \quad & \rho(va) = v_{[0]} \otimes v_{[1]}a \end{aligned}$$

for all $v \in V$ and $a \in A$. The category of right $A \otimes A$ -comodules is denoted by $\mathcal{M}^{A \otimes A}$. There is an adjunction between \mathcal{M}_k and $\mathcal{M}^{A \otimes A}$. Cipolla's descent data are nothing else than $A \otimes A$ -comodules, and Cipolla's version of the Faithfully Flat Descent Theorem asserts that this is a pair of inverse equivalences if A is faithfully flat over k , we refer to [9] for a detailed discussion.

First observe that this machinery works for a general extension $k \rightarrow A$ of rings, that is, A and k are not necessarily commutative. In this note, however, we keep k commutative. If A is commutative, then the categories $\underline{\text{Desc}}(A/k)$ and $\mathcal{M}^{A \otimes A}$ are isomorphic. $(V, g) \in \underline{\text{Desc}}(A/k)$ corresponds to $(V, \rho) \in \mathcal{M}^{A \otimes A}$, with $\rho(v) = g(1 \otimes v)$.

Sometimes it is argued that this generalization is not satisfactory, since there is no counterpart to Proposition 1.1 in the case where A is noncommutative. In this note, we will present an appropriate generalization $\underline{\text{Desc}}(A/k)$ to the noncommutative situation, with a suitable generalized version of Proposition 1.1, see Proposition 2.5 and Remark 2.9.

2. THE CENTER OF THE CATEGORY OF BIMODULES

Throughout, A is an algebra over a commutative ring k .

Definition 2.1. A right Yetter-Drinfeld A^e -module consists of a pair (V, ρ) , such that V is an A -bimodule, $(V, \rho) \in \mathcal{M}^{A \otimes A}$ and the following compatibility conditions hold:

$$\begin{aligned} (5) \quad & \rho(av) = v_{[0]} \otimes av_{[1]}; \\ (6) \quad & a\rho(v) = av_{[0]} \otimes v_{[1]} = v_{[0]}a \otimes v_{[1]}. \end{aligned}$$

A morphism $(V, \rho) \rightarrow (V', \rho')$ of Yetter-Drinfeld modules is a map $f : V \rightarrow V'$ that is an A -bimodule and $A^{(2)}$ -comodule map. The category of Yetter-Drinfeld modules will be denoted by \mathcal{YD}^{A^e} .

Take $(V, \rho) \in \mathcal{YD}^{A^e}$. Then

$$(7) \quad av \stackrel{(2)}{=} (av)_{[0]}(av)_{[1]} \stackrel{(5)}{=} v_{[0]}av_{[1]},$$

and

$$(8) \quad v_{[1]}v_{[0]} \stackrel{(7)}{=} v_{[0][0]}v_{[1]}v_{[0][1]} \stackrel{(3)}{=} v_{[0]}v_{[1]} \stackrel{(2)}{=} v.$$

Proposition 2.2. *The forgetful functor $U : \mathcal{YD}^{A^e} \rightarrow \mathcal{M}^{A \otimes A}$ is an isomorphism of categories.*

Proof. We define a functor $P : \mathcal{M}^{A \otimes A} \rightarrow \mathcal{YD}^{A^e}$. For $V \in \mathcal{M}^{A \otimes A}$, let $P(V) = V$ as an $A^{(2)}$ -comodule, with left A -action defined by $av = v_{[0]}av_{[1]}$. Then

$$\rho(av) = \rho(v_{[0]}av_{[1]}) \stackrel{(4)}{=} \rho(v_{[0]})av_{[1]} \stackrel{(3)}{=} v_{[0]} \otimes av_{[1]},$$

and (5) is satisfied. The left A -action is associative since

$$b(av) = (av)_{[0]}b(av)_{[1]} \stackrel{(5)}{=} v_{[0]}bav_{[1]} = (ba)v.$$

Finally we show that (6) holds:

$$av_{[0]} \otimes v_{[1]} = v_{[0][0]}av_{[0][1]} \otimes v_{[1]} \stackrel{(3)}{=} v_{[0]}a \otimes v_{[1]}.$$

This shows that $P(V) \in \mathcal{YD}^{A^e}$. If $f : V \rightarrow W$ is a morphism in $\mathcal{M}^{A \otimes A}$, then it is also a morphism $P(V) \rightarrow P(W)$ in \mathcal{YD}^{A^e} . To this end, we need to show that f is left A -linear:

$$f(av) = f(v_{[0]}av_{[1]}) = f(v_{[0]})av_{[1]} = f(v)_{[0]}af(v)_{[1]} = af(v).$$

We used the fact that f is right A -linear and right $A \otimes A$ -colinear. Finally, it is clear that the functors P and V are inverses. \square

Recall from Section 1.1 that $\mathcal{W}_r({}_A\mathcal{M}_A)$ is the weak right center of the monoidal category $({}_A\mathcal{M}_A, - \otimes_A -, A)$ of A -bimodules.

Proposition 2.3. *The categories $\mathcal{W}_r({}_A\mathcal{M}_A)$ and \mathcal{YD}^{A^e} are isomorphic.*

Proof. Let $(V, c_{-,V})$ be an object of $\mathcal{W}_r({}_A\mathcal{M}_A)$. For every A -bimodule M , we have an A -bimodule map $c_{M,V} : M \otimes_A V \rightarrow V \otimes_A M$, which is natural in M . Consider

$$g = c_{A \otimes A, V} : A^{(2)} \otimes_A V \cong A \otimes V \rightarrow V \otimes_A A^{(2)} \cong V \otimes A,$$

and define $\rho : V \rightarrow V \otimes A$ by $\rho(v) = g(1 \otimes v) = v_{[0]} \otimes v_{[1]} \in V \otimes A$. $c_{-,V}$ is then completely determined by ρ : for $m \in M$, define the A -bimodule map $f_m : A^{(2)} \rightarrow M$ by the formula $f_m(a \otimes b) = amb$. From the naturality of $c_{-,V}$, it follows that we have a commutative diagram

$$\begin{array}{ccc} A^{(2)} \otimes_A V & \xrightarrow{g} & V \otimes_A A^{(2)} \\ f_m \otimes_A V \downarrow & & \downarrow V \otimes_A f_m \\ M \otimes_A V & \xrightarrow{c_{M,V}} & V \otimes_A M \end{array}$$

Evaluating the diagram at $1 \otimes v$, we find

$$(9) \quad c_{M,V}(m \otimes_A v) = v_{[0]} \otimes_A m v_{[1]}.$$

We will now show that $(V, \rho) \in \mathcal{YD}^{A^e}$. Using the fact that $c_{M,V}$ is right A -linear, well-defined and left A -linear, we find

$$\begin{aligned} (va)_{[0]} \otimes m(va)_{[1]} &= c_{M,V}(m \otimes_A va) = c_{M,V}(m \otimes_A v)a = v_{[0]} \otimes_A mv_{[1]}a; \\ v_{[0]} \otimes_A mav_{[1]} &= c_{M,V}(ma \otimes_A v) = c_{M,V}(m \otimes_A av) = (av)_{[0]} \otimes m(av)_{[1]}; \\ v_{[0]} \otimes_A amv_{[1]} &= c_{M,V}(am \otimes_A v) = ac_{M,V}(m \otimes_A v) = av_{[0]} \otimes_A mv_{[1]}. \end{aligned}$$

If we take $M = A^{(2)}$ and $m = 1 \otimes 1$ in these formulas, we obtain (4), (5) and (6). $c_{A,V}$ is the canonical isomorphism $A \otimes_A V \rightarrow V \otimes_A A$, hence $v \otimes_A 1 = c_{A,V}(1 \otimes_A v) = v_{[0]} \otimes_A v_{[1]}$, and (2) follows. Finally, we have the commutative diagram

$$\begin{array}{ccc} M \otimes_A N \otimes_A V & \xrightarrow{c_{M \otimes_A N, V}} & V \otimes_A M \otimes_A N \\ & \searrow M \otimes_A c_{N, V} \quad \nearrow c_{M, V} \otimes_A N & \\ & M \otimes_A V \otimes_A N & \end{array}$$

We evaluate the diagram at $m \otimes_A n \otimes_A v$:

$$\begin{aligned} v_{[0]} \otimes_A m \otimes_A nv_{[1]} &= c_{M \otimes_A N, V}(m \otimes_A n \otimes_A v) \\ &= ((c_{M, V} \otimes_A N) \circ (M \otimes_A c_{N, V}))(m \otimes_A n \otimes_A v) \\ &= (c_{M, V} \otimes_A N)(m \otimes_A v_{[0]} \otimes_A nv_{[1]}) = v_{[0][0]} \otimes_A mv_{[0][1]} \otimes_A Anv_{[1]} \end{aligned}$$

(3) follows after we take $M = N = A^{(2)}$ and $m = n = 1 \otimes 1$.

Conversely, given $(V, \rho) \in \mathcal{YD}^{A^e}$, we define $c_{-,V}$ using (9). Straightforward computations show that $(V, c_{-,V}) \in \mathcal{W}_r(A\mathcal{M}_A)$. \square

Remark 2.4. It is well-known that $A^e = A \otimes A^{\text{op}}$ is an A -bialgebroid. The arguments in Proposition 2.3 can be generalized, leading to a description of the (weak) center of the category of modules over a bialgebroid, and to the definition of Yetter-Drinfeld module over a bialgebroid, see [20]. In fact, the Yetter-Drinfeld modules of Definition 2.1 are precisely the Yetter-Drinfeld modules over the bialgebroid A^e , justifying our terminology.

Our next aim is to show that condition (2) in Definition 2.1 can be replaced by the condition that g is invertible.

Proposition 2.5. *Let A be a k -algebra, and assume that $\rho : V \rightarrow V \otimes A$ satisfies (3-6). Then (2) holds if and only if $g : A \otimes V \rightarrow V \otimes A$, $g(a \otimes v) = av_{[0]} \otimes v_{[1]}$ is invertible.*

Proof. Assume that (2) holds. For all $a \in A$ and $v \in V$, we have

$$\begin{aligned} (\tau \circ g \circ \tau \circ g)(a \otimes v) &= (\tau \circ g)(v_{[1]} \otimes av_{[0]}) \\ &= \tau(v_{[1]}(av_{[0]})_{[0]} \otimes (av_{[0]})_{[1]}) \stackrel{(5)}{=} \tau(v_{[1]}v_{[0][0]} \otimes av_{[0][1]}) \\ &\stackrel{(6)}{=} \tau(v_{[0][0]}v_{[1]} \otimes av_{[0][1]}) \stackrel{(3)}{=} \tau(v_{[0]}v_{[1]} \otimes a) \stackrel{(2)}{=} a \otimes v. \end{aligned}$$

We conclude that $\tau \circ g \circ \tau \circ g = \text{Id}_{A \otimes V}$. Composing to the left and to the right with the switch map τ , we find $g \circ \tau \circ g \circ \tau = \text{Id}_{V \otimes A}$. Thus $g^{-1} = \tau \circ g \circ \tau$.

Conversely, assume that g is invertible. For any $v \in V$ we have:

$$g(1 \otimes v_{[0]}v_{[1]}) = \rho(v_{[0]}v_{[1]}) \stackrel{(4)}{=} \rho(v_{[0]})v_{[1]} \stackrel{(3)}{=} v_{[0]} \otimes v_{[1]} = g(1 \otimes v).$$

(2) follows after we apply g^{-1} to both sides and multiply the two tensor factors. \square

Proposition 2.6. *The (right) center of the category of A -bimodules coincides with its (right) weak center: $\mathcal{Z}_r({}_A\mathcal{M}_A) = \mathcal{W}_r({}_A\mathcal{M}_A)$.*

Proof. Take $(V, c_{-,V}) \in \mathcal{W}_r({}_A\mathcal{M}_A)$. We will show that $c_{M,V}$ is invertible, for every A -bimodule M . Let g and ρ be as in Proposition 2.3. We claim that

$$(10) \quad c_{M,V}^{-1}(v \otimes_A m) = v_{[1]}m \otimes_A v_{[0]}.$$

Indeed, for all $m \in M$ and $v \in V$, we have that

$$\begin{aligned} (c_{M,V}^{-1} \circ c_{M,V})(m \otimes_A v) &\stackrel{(9,10)}{=} v_{[0][1]}mv_{[0][0]} \otimes_A v_{[0]} \stackrel{(3)}{=} m \otimes_A v_{[1]}v_{[0]} \stackrel{(8)}{=} m \otimes_A v; \\ (c_{M,V} \circ c_{M,V}^{-1})(v \otimes_A m) &\stackrel{(10,9)}{=} v_{[0][0]} \otimes_A v_{[1]}mv_{[0][1]} \stackrel{(3)}{=} v_{[0]}v_{[1]} \otimes_A m \stackrel{(2)}{=} v \otimes_A m. \end{aligned}$$

□

If V and W are A -bimodules, then $V \otimes W$ is an $A^{(2)}$ -bimodule. Consider a map $g : A \otimes V \rightarrow V \otimes A$ in ${}_{A^{(2)}}\mathcal{M}_{A^{(2)}}$. The maps g_1, g_2, g_3 defined by (1) are in ${}_{A^{(3)}}\mathcal{M}_{A^{(3)}}$.

Definition 2.7. Let A be a k -algebra. A descent datum consists of an A -bimodule V together with an $A^{(2)}$ -bimodule map $g : A \otimes V \rightarrow V \otimes A$ such that $g_2 = g_3 \circ g_1$ and $(\psi \circ g)(a \otimes v) = v$, for all $v \in V$, where ψ is the map $V \otimes A \rightarrow A$, $\psi(v \otimes a) = va$. A morphism between two descent data (V, g) and (V', g') is an A -bimodule map $f : V \rightarrow V'$ such that $(f \otimes A) \circ g = g' \circ (A \otimes f)$. The category of descent data is denoted by $\underline{\text{Desc}}(A/k)$.

Proposition 2.8. *The categories $\underline{\text{Desc}}(A/k)$ and \mathcal{YD}^{A^e} are isomorphic.*

Proof. Let $(V, \rho) \in \mathcal{YD}^{A^e}$, and define $g : A \otimes V \rightarrow V \otimes A$ by $g(a \otimes v) = av_{[0]} \otimes v_{[1]}$. First we show that g is an $A^{(2)}$ -bimodule map.

$$\begin{aligned} g(ba \otimes cv) &= ba(cv)_{[0]} \otimes (cv)_{[1]} \stackrel{(5)}{=} bav_{[0]} \otimes cv_{[1]} = (b \otimes c)g(a \otimes v); \\ g(ab \otimes vc) &= ab(vc)_{[0]} \otimes (vc)_{[1]} \stackrel{(4)}{=} abv_{[0]} \otimes v_{[1]}c \stackrel{(6)}{=} av_{[0]}b \otimes v_{[1]}c = g(a \otimes v)(b \otimes c). \end{aligned}$$

Now $g_3 \circ g_1 = g_2$ since

$$\begin{aligned} (g_3 \circ g_1)(a \otimes b \otimes v) &= g_3(a \otimes bv_{[0]} \otimes v_{[1]}) = a(bv_{[0]})_{[0]} \otimes (bv_{[0]})_{[1]} \otimes v_{[1]} \\ &\stackrel{(5)}{=} av_{[0][0]} \otimes bv_{[0][1]} \otimes v_{[1]} \stackrel{(3)}{=} av_{[0]} \otimes b \otimes v_{[1]} = g_2(a \otimes b \otimes v). \end{aligned}$$

Finally, $(m \circ g)(1 \otimes v) = v_{[0]}v_{[1]} = v$, and we conclude that $(V, g) \in \underline{\text{Desc}}(A/k)$.

Conversely, let $(V, g) \in \underline{\text{Desc}}(A/k)$, and define $\rho : V \rightarrow V \otimes A$ by $\rho(v) = g(1 \otimes v)$. Then $f(a \otimes v) = a\rho(v) = av_{[0]} \otimes v_{[1]}$. It is easy to show that (2) and (4-6) are satisfied:

$$\begin{aligned} v &= (m \circ g)(1 \otimes v) = m(\rho(v)) = v_{[0]}v_{[1]}; \\ \rho(va) &= g(1 \otimes va) = g(1 \otimes v)(1 \otimes a) = v_{[0]} \otimes v_{[1]}a; \\ \rho(av) &= g(1 \otimes av) = (1 \otimes a)g(1 \otimes v) = v_{[0]} \otimes av_{[1]}; \\ a\rho(v) &= (a \otimes 1)g(v) = g(a \otimes v) = g(1 \otimes v)(a \otimes 1) = v_{[0]}a \otimes v_{[1]}. \end{aligned}$$

We have already computed $g_3 \circ g_1$ and g_2 . This computation stays valid, since we only used (5), which holds. Expressing that $(g_3 \circ g_1)(1 \otimes 1 \otimes v) = g_2(1 \otimes 1 \otimes v)$, we find (3). We conclude that $(V, \rho) \in \mathcal{YD}^{A^e}$. □

Remarks 2.9. 1. It follows from the proof of Proposition 2.8 that the definition of a descent datum can be restated as follows: $V \in {}_A\mathcal{M}_A$, an invertible map $g : A \otimes V \rightarrow V \otimes A$ in ${}_{A^{(2)}}\mathcal{M}_{A^{(2)}}$ satisfying $g_2 = g_3 \circ g_1$.

2. We look at the particular case where A is commutative. Take $(V, g) \in \underline{\text{Desc}}(A/k)$ and let (V, ρ) be the corresponding object of \mathcal{YD}^{A^e} . Then we know that $av = v_{[0]}av_{[1]} = v_{[0]}v_{[1]}a = va$, hence the left A -action on V coincides with the right A -action. Consequently, the left and right $A^{(2)}$ -actions on $A \otimes V$ and $V \otimes A$ coincide. So we can view a descent datum (V, g) as a right A -module V together with a right $A^{(2)}$ -linear map $g : A \otimes V \rightarrow V \otimes A$ satisfying $g_2 = g_3 \circ g_1$ and $(\psi \circ g)(1 \otimes v) = v$, or, equivalently, g invertible. These are precisely the descent data [15] that we discussed in Section 1.2.

The main results of this paper are summarized as follows:

Theorem 2.10. *For a k -algebra A , the categories $\underline{\text{Desc}}(A/k)$, \mathcal{YD}^{A^e} , $\mathcal{M}^{A \otimes A}$, $\mathcal{W}_r({}_A\mathcal{M}_A)$ and $\mathcal{Z}_r({}_A\mathcal{M}_A)$ are isomorphic. If A is faithfully flat over k then these isomorphic categories are equivalent to the category of k -modules.¹*

$\mathcal{Z}_r({}_A\mathcal{M}_A)$ is a braided monoidal category, hence we can define braided monoidal structures on the five isomorphic categories in Theorem 2.10. In particular, the category of comodules over the Sweedler canonical A -coring $A \otimes A$ is braided monoidal. Explicitly we have:

Corollary 2.11. *Let A be a k -algebra. Then $(\mathcal{M}^{A \otimes A}, - \otimes_A -, A)$ is a braided monoidal category as follows: for $V \in \mathcal{M}^{A \otimes A}$, we have a left A -action on V defined by $a \cdot v = v_{[0]}av_{[1]}$. The tensor product is then just the tensor product over A , and the coaction on $V \otimes_A V'$ is given by the formula $\rho(v \otimes_A v') = v_{[0]} \otimes_A v'_{[0]} \otimes v_{[1]}v'_{[1]}$. The unit is A , with $A \otimes A$ -coaction $\rho(a) = 1 \otimes a$. The left A -action on A then coincides with the left regular representation: $b \cdot a = a_{[0]}ba_{[1]} = ba$. The braiding c on $\mathcal{M}^{A \otimes A}$ is given by*

$$c_{V', V}(v' \otimes_A v) = v_{[0]} \otimes_A v'v_{[1]} ; c_{V', V}^{-1}(v \otimes_A v') = v_{[1]}v' \otimes_A v_{[0]}.$$

Proof. This follows of course from the general theory of the center construction, but all axioms can be easily verified directly. \square

Remark 2.12. An interesting interpretation of Theorem 2.10 and Corollary 2.11 was communicated to us by T. Brzezinski. In [6], it was observed that there is a close relationship between corings with a grouplike element and noncommutative differential geometry. One of the results in this direction is the following: the category $\mathcal{M}^{A \otimes A}$ is isomorphic to the category $\mathbf{Conn}(A/k, \text{Omega}(A \otimes A/k))$ of right A -modules with a flat connection, see [6, Theorem 5] or [8, Sec. 29]. It then follows from Corollary 2.11 that $\mathbf{Conn}(A/k, \text{Omega}(A \otimes A/k))$ is a braided monoidal category. In the forthcoming [7], the braiding on $\mathcal{M}^{A \otimes A}$ is applied to prove that any flat connection in a right A -module is an A -bimodule connection.

¹The fact that $\mathcal{Z}_r({}_A\mathcal{M}_A)$ is equivalent to the category of k -modules if A is faithfully flat can be also derived from [21, Theorem 3.3].

3. FINITELY GENERATED PROJECTIVE ALGEBRAS

Now we focus attention to the case where A is finitely generated and projective as a k -module, which means that the k -linear map

$$(11) \quad \varphi : A^* \otimes A \rightarrow \mathcal{A} = \text{End}_k(A), \quad \varphi(a^* \otimes b)(x) = \langle a^*, x \rangle b$$

is an isomorphism. Then $\varphi^{-1}(\text{Id}_A) = \sum_i a_i^* \otimes a_i$ is called a finite dual basis of A , and is characterized by the formula $\sum_i \langle a_i^*, x \rangle a_i = x$, for all $x \in A$. In this situation, we also have that

$$(12) \quad \varphi^{-1}(f) = \sum_i a_i^* \otimes f(a_i),$$

for all $f \in \mathcal{A}$. Indeed, $\varphi(\sum_i a_i^* \otimes f(a_i))(x) = \sum_i \langle a_i^*, x \rangle f(a_i) = f(x)$, for all $x \in A$. Recall that we also have an algebra map $F : A \otimes A^{\text{op}} \rightarrow \text{End}_k(A)$, $F(a \otimes b)(x) = axb$. It is then easy to show that

$$(13) \quad \varphi(a^* \otimes a) = F(a \otimes 1) \circ \varphi(a^* \otimes 1) = F(1 \otimes a) \circ \varphi(a^* \otimes 1).$$

The categories $\mathcal{M}^{A \otimes A}$ and ${}_{\mathcal{A}}\mathcal{M}$ are isomorphic. If V is a right $A \otimes A$ -comodule, then we have a left \mathcal{A} -action given by

$$(14) \quad f \cdot v = v_{[0]} f(v_{[1]}).$$

for all $f \in \mathcal{A} = \text{End}_k(A)$ and $v \in V$. Conversely, for $V \in {}_{\mathcal{A}}\mathcal{M}$, we have a right $A \otimes A$ -coaction now given by

$$(15) \quad \rho(v) = \sum_i f_i \cdot v \otimes a_i,$$

where we write $f_i = \varphi(a_i^* \otimes 1)$. This is well-known and can be verified easily. It also has an explanation in terms of corings: the left dual of the A -coring $A \otimes A$ is ${}_{\mathcal{A}}\text{Hom}(A \otimes A, A) \cong \text{End}(A)^{\text{op}}$ as A -rings, see for example [8]. We will now transport the braided monoidal structure of $\mathcal{M}^{A \otimes A}$ to ${}_{\mathcal{A}}\mathcal{M}$.

If $V \in {}_{\mathcal{A}}\mathcal{M}$, then $V \in {}_{\mathcal{A}}\mathcal{M}_A$, by restriction of scalars via F . Now we also have that $V \in \mathcal{M}^{A \otimes A} \cong \mathcal{YD}^{A^e}$, and this gives a second A -bimodule structure on V . These two bimodule structures coincide:

$$\begin{aligned} F(1 \otimes a) \cdot v &\stackrel{(14)}{=} v_{[0]}(F(1 \otimes a)(v_{[1]})) = v_{[0]}v_{[1]}a = va; \\ F(a \otimes 1) \cdot v &\stackrel{(14)}{=} v_{[0]}(F(a \otimes 1)(v_{[1]})) = v_{[0]}av_{[1]} = av. \end{aligned}$$

Now take $V, W \in {}_{\mathcal{A}}\mathcal{M}$. Then $V \otimes_A W \in \mathcal{M}^{A \otimes A} \cong {}_{\mathcal{A}}\mathcal{M}$. We describe the \mathcal{A} -action on $V \otimes_A W$.

$$\begin{aligned} f \cdot (v \otimes_A w) &\stackrel{(14)}{=} v_{[0]} \otimes_A w_{[0]} f(v_{[1]} w_{[1]}) \stackrel{(15)}{=} \sum_{i,j} f_i \cdot v \otimes_A (f_j \cdot w) f(a_i a_j) \\ &= \sum_{i,j} f_i \cdot v \otimes_A (F(1 \otimes f(a_i a_j)) \circ \varphi(a_j^* \otimes 1)) \cdot w \\ &\stackrel{(13)}{=} \sum_{i,j} f_i \cdot v \otimes_A \varphi(a_j^* \otimes f(a_i a_j)) \cdot w \stackrel{(12)}{=} \sum_i f_i \cdot v \otimes_A f(a_i -) \cdot w, \end{aligned}$$

where $f(a-) \in \mathcal{A}$ is the map sending $x \in A$ to $f(ax)$; we have an alternative description:

$$\begin{aligned}
f \cdot (v \otimes_A w) &= \sum_{i,j} f_i \cdot v \otimes_A (F(1 \otimes f(a_i a_j)) \circ \varphi(a_j^* \otimes 1)) \cdot w \\
&\stackrel{(13)}{=} \sum_{i,j} f_i \cdot v \otimes_A (F(f(a_i a_j) \otimes 1) \circ \varphi(a_j^* \otimes 1)) \cdot w \\
&= \sum_{i,j} (F(1 \otimes f(a_i a_j)) \circ \varphi(a_i^* \otimes 1)) \cdot v \otimes_A f_j \cdot w \\
&\stackrel{(13)}{=} \varphi(a_i^* \otimes f(a_i a_j)) \cdot v \otimes_A f_j \cdot w \stackrel{(12)}{=} \sum_j f(-a_j) \cdot v \otimes_A f_j \cdot w.
\end{aligned}$$

The braiding is given by the formula $c_{V,W}(v \otimes_A w) = w_{[0]} \otimes_A v w_{[1]} = \sum_i f_i \cdot w \otimes v a_i$. We summarize our results:

Proposition 3.1. *Let A be a finitely generated projective k -algebra, with finite dual basis $\sum_i a_i^* \otimes a_i$, and write $f_i = \varphi(a_i^* \otimes 1)$. The category of left $\text{End}_k(A)$ -modules is a braided monoidal category. The tensor product is the tensor product over A ; a left $\text{End}_k(A)$ -module is an A -bimodule by restriction of scalars via F . The left $\text{End}_k(A)$ -action on $V \otimes_A W$ is given by*

$$f \cdot (v \otimes_A w) = \sum_i f_i \cdot v \otimes_A f(a_i -) \cdot w = \sum_j f(-a_j) \cdot v \otimes_A f_j \cdot w$$

for all $f \in \text{End}_k(A)$, $v \in V$ and $w \in W$. The unit object is A , with its obvious left $\text{End}_k(A)$ -action $f \cdot a = f(a)$. The braiding is given by $c_{V,W}(v \otimes_A w) = \sum_i f_i \cdot w \otimes v a_i$.

Remark 3.2. As we mentioned in the introduction, the category of Yetter-Drinfeld modules over a finite Hopf algebra is isomorphic to the category of modules over the Drinfeld double. We have an analogous result here: if A is finite (that is, finitely generated projective), then the category of Yetter-Drinfeld A^e -modules is isomorphic to the category of representations of $\text{End}_k(A)$. In fact, this tells us that we can consider $\text{End}_k(A)$ as the Drinfeld double of A^e .

Example 3.3. Let $A = k^n = \bigoplus_{i=1}^n k e_i$, with multiplication $e_i e_j = \delta_{ij} e_i$ and unit $1 = \sum_{i=1}^n e_i$. Let $e_i^* \in A^*$ be given by $\langle e_i^*, e_j \rangle = \delta_{ij}$. We can then identify $M_n(k)$ and $\text{End}_k(A)$, where an endomorphism of A corresponds to its matrix with respect to the basis $\{e_1, \dots, e_n\}$. It is then easy to see that $\varphi(e_i^* \otimes e_j) = e_{ji}$, the elementary matrix with 1 in the (i, j) -position and 0 elsewhere. Now we easily compute that $f_l = \varphi(\sum_r e_l^* \otimes e_r) = \sum_r e_{rl}$, $e_{ii} = F(e_i \otimes 1) = F(1 \otimes e_i)$ and $e_{ij}(e_l -) = \delta_{jl} e_{ij}$. Let V and W be left $M_n(k)$ -modules. Then $V \otimes_{k^n} W$ is again a left $M_n(k)$ -module, the left $M_n(k)$ -action is given by the formulas in Proposition 3.1, which simplify as follows:

$$\begin{aligned}
e_{ij} \cdot (v \otimes_{k^n} w) &= \sum_{l,r} e_{rl} \cdot v \otimes_{k^n} \delta_{jl} e_{ij} \cdot w = \sum_r e_{rj} \cdot v \otimes_{k^n} e_{ij} \cdot w \\
&= \sum_r e_{rj} \cdot v \otimes_{k^n} (e_{ii} e_{ij}) \cdot w = \sum_r e_{rj} \cdot v \otimes_{k^n} e_i (e_{ij} \cdot w) \\
&= \sum_r (e_{rj} \cdot v) e_i \otimes_{k^n} e_{ij} \cdot w = \sum_r (e_{ii} e_{rj}) \cdot v \otimes_{k^n} e_{ij} \cdot w
\end{aligned}$$

$$= e_{ij} \cdot v \otimes_{k^n} e_{ij} \cdot w.$$

Finally, we compute the braiding

$$\begin{aligned} c_{V,W}(v \otimes_{k^n} w) &= \sum_i f_i \cdot w \otimes_{k^n} v e_i = \sum_{i,r} e_{ri} \cdot w \otimes_{k^n} e_i v \\ &= \sum_{i,r} (e_{ri} \cdot w) e_i \otimes_{k^n} v = \sum_{i,r} (e_{ii} e_{ri}) \cdot w \otimes_{k^n} v \\ &= \sum_i e_{ii} \cdot w \otimes_{k^n} v = w \otimes_{k^n} v. \end{aligned}$$

The fact that the representation category of a matrix algebra is monoidal can also be understood in a completely different way. Weak bialgebras and Hopf algebras were introduced in [4]. The representation category of a weak bialgebra is monoidal, see [22, 19, 3]. The tensor is the tensor product over $H_t = \text{Im } \varepsilon_t$, where $\varepsilon_t : H \rightarrow H$ is given by the formula $\varepsilon_t(h) = \langle \varepsilon, 1_{(1)} h \rangle 1_{(2)}$. $H = M^n(k)$ is a weak Hopf algebra, with comultiplication and counit given by the formulas $\Delta(e_{ij}) = e_{ij} \otimes e_{ij}$ and $\varepsilon(e_{ij}) = 1$. In fact it is a groupoid algebra, over the groupoid with n objects, and precisely one morphism e_{ij} between the objects i and j . In this situation, it is easy to show that $\Delta(1) = \sum_l \Delta(e_{ll}) = \sum_l e_{ll} \otimes e_{ll}$, and $\varepsilon_t(e_{ij}) = \sum_l \langle \varepsilon, e_{ll} e_{ij} \rangle e_{ll} = e_{ii}$, so that $H_t = \oplus_i k e_{ii} \cong k^n$. The monoidal structure on $M_n(k)$ then coincides with the one that we found above. The braiding comes from a quasitriangular structure on $M_n(k)$.

4. APPLICATION TO THE QUANTUM YANG-BAXTER EQUATION

Our results lead to the construction of a new family of solutions of the quantum Yang-Baxter equation. More precisely, to every object of $\mathcal{YD}^{A^e} \cong \mathcal{M}^{A \otimes A}$, we can associate a solution of the quantum Yang-Baxter equation.

Theorem 4.1. *Let A be a k -algebra and $(V, \rho) \in \mathcal{YD}^{A^e}$. Then*

$$(16) \quad \Omega = \Omega_{(V, \rho)} : V \otimes V \rightarrow V \otimes V, \quad \Omega(v \otimes w) = w_{[0]} \otimes w_{[1]} v,$$

is a solution of the quantum Yang-Baxter equation $\Omega^{12} \Omega^{13} \Omega^{23} = \Omega^{23} \Omega^{13} \Omega^{12}$ in $\text{End}(V^{(3)})$. In particular, if $(V, \rho) \in \mathcal{M}^{A \otimes A}$, then

$$(17) \quad \Omega = \Omega_{(V, \rho)} : V \otimes V \rightarrow V \otimes V, \quad \Omega(v \otimes w) = w_{[0]} \otimes v_{[0]} w_{[1]} v_{[1]},$$

is a solution of the quantum Yang-Baxter equation and $\Omega^3 = \Omega$ in $\text{End}(V^{(2)})$.

Proof. For all $v, w, t \in V$, we have that:

$$\begin{aligned} \Omega^{12} \Omega^{13} \Omega^{23}(v \otimes w \otimes t) &= \Omega^{12} \Omega^{13}(v \otimes t_{[0]} \otimes t_{[1]} w) \\ &\stackrel{(5)}{=} \Omega^{12}((t_{[1]} w)_{[0]} \otimes t_{[0]} \otimes (t_{[1]} w)_{[1]} v) \stackrel{(3)}{=} t_{[0][0]} \otimes t_{[0][1]} (t_{[1]} w)_{[0]} \otimes (t_{[1]} w)_{[1]} v \\ &\stackrel{(5)}{=} t_{[0][0]} \otimes t_{[0][1]} t_{[1]} w_{[0]} \otimes w_{[1]} v \stackrel{(3)}{=} t_{[0]} \otimes t_{[1]} w_{[0]} \otimes w_{[1]} v; \\ \Omega^{23} \Omega^{13} \Omega^{12}(v \otimes w \otimes t) &= \Omega^{23} \Omega^{13}(w_{[0]} \otimes w_{[1]} v \otimes t) \\ &\stackrel{(5)}{=} \Omega^{23}(t_{[0]} \otimes w_{[1]} v \otimes t_{[1]} w_{[0]}) \stackrel{(3)}{=} t_{[0]} \otimes (t_{[1]} w_{[0]})_{[0]} \otimes (t_{[1]} w_{[0]})_{[1]} w_{[1]} v \\ &\stackrel{(5)}{=} t_{[0]} \otimes t_{[1]} w_{[0][0]} \otimes w_{[0][1]} w_{[1]} v \stackrel{(3)}{=} t_{[0]} \otimes t_{[1]} w_{[0]} \otimes w_{[1]} v \end{aligned}$$

Thus $\Omega^{12} \Omega^{13} \Omega^{23} = \Omega^{23} \Omega^{13} \Omega^{12}$. We have seen in Proposition 2.2 that $(V, \rho) \in \mathcal{M}^{A \otimes A}$, with left A -action $a \cdot v = v_{[0]} a v_{[1]}$, is an object of \mathcal{YD}^{A^e} . With this identification the canonical map (16) takes precisely the form (17). Now, for all $v, w \in V$ we have:

$$\begin{aligned} \Omega^2(v \otimes w) & \stackrel{(3)}{=} \Omega(w_{[0]} \otimes v_{[0]} w_{[1]} v_{[1]}) \stackrel{(4)}{=} v_{[0][0]} \otimes w_{[0][0]} v_{[0][1]} w_{[1]} v_{[1]} w_{[0][1]} \\ & \stackrel{(3)}{=} v_{[0]} \otimes w_{[0]} w_{[1]} v_{[1]} \stackrel{(2)}{=} v_{[0]} \otimes w v_{[1]}; \\ \Omega^3(v \otimes w) & \stackrel{(3)}{=} \Omega(v_{[0]} \otimes w v_{[1]}) \stackrel{(4)}{=} w_{[0]} \otimes v_{[0][0]} w_{[1]} v_{[1]} v_{[0][1]} \\ & \stackrel{(3)}{=} w_{[0]} \otimes v_{[0]} w_{[1]} v_{[1]} = \Omega(v \otimes w). \end{aligned}$$

□

It is well-known that $x = x^1 \otimes x^2 \in A^{(2)}$ is grouplike if and only if $x^1 x^2 = 1$ and $X^1 \otimes X^2 x^1 \otimes x^2 = X^1 \otimes 1 \otimes X^2$. Grouplike elements of a coring \mathcal{C} are in bijective correspondence to right \mathcal{C} -coactions on A . In the case where $\mathcal{C} = A^{(2)}$, the right $A \otimes A$ -coaction on A associated to x is $\rho(a) = x^1 \otimes x^2 a$. For $M \in \mathcal{M}^{A \otimes A}$, we define

$$M^{\text{cox}} = \{m \in M \mid \rho(m) = m x^1 \otimes x^2\}.$$

A^{cox} is a subalgebra of A . Suppose that we have an algebra morphism $i : B \rightarrow A^{\text{cox}}$. Then we have a pair of adjoint functors (see [9, Sec. 1]) $(- \otimes_B A, (-)^{\text{cox}})$ between the categories \mathcal{M}_B and $\mathcal{M}^{A \otimes A}$. The right coaction on $N \otimes_B A$ is simply $\rho(n \otimes_B a) = n \otimes_B x^1 \otimes x^2 a$. This construction allows us to give examples of $A \otimes A$ -comodules, and, a fortiori, solutions of the quantum Yang-Baxter equation, applying Theorem 4.1. We then obtain the following.

Proposition 4.2. *Let x be a grouplike element of $A \otimes A$, and let $i : B \rightarrow A^{\text{cox}}$ be an algebra morphism. For $N \in \mathcal{M}_B$, the map $\Omega : (N \otimes_B A)^{(2)} \rightarrow (N \otimes_B A)^{(2)}$ given by*

$$(18) \quad \Omega((n \otimes_B a) \otimes (m \otimes_B b)) = (m \otimes_B x^1) \otimes (n \otimes_B X^1 x^2 b X^2 a)$$

is a solution of the quantum Yang-Baxter equation.

As a particular example, we can take $x = 1 \otimes 1$, $B = k$, $N \in \mathcal{M}_k$. Then (18) takes the form $\Omega(m \otimes a \otimes n \otimes b) = n \otimes 1 \otimes m \otimes ba$. In particular, if we take $N = k$, then $\Omega(a \otimes b) = 1 \otimes ba$, and this shows that Ω is not necessarily bijective.

We now present another way to construct comodules over $A \otimes A$. It is shown in [1] that there is a bijective correspondence between braidings on the category of A -bimodules and elements $R \in A^{(3)}$ satisfying the conditions

$$(19) \quad R^1 \otimes a R^2 \otimes R^3 = R^1 \otimes R^2 \otimes R^3 a$$

$$(20) \quad R^1 R^2 \otimes R^3 = R^2 \otimes R^3 R^1 = 1 \otimes 1,$$

see [1, Theorem 2.4]. We then say that (A^e, R) is quasitriangular, and we call R an R -matrix. R satisfies several other equations, we mention that R is invariant under cyclic permutation of the tensor factors, and

$$(21) \quad R^1 \otimes R^2 \otimes 1 \otimes R^3 = r^1 R^1 \otimes r^2 \otimes r^3 R^2 \otimes R^3,$$

see [1, Theorem 2.4]. Yetter-Drinfeld modules can be constructed from bimodules over quasitriangular algebras as follows.

Proposition 4.3. *Let A be a k -algebra, let V be an A -bimodule, and let $R \in A^{(3)}$ be an R -matrix. Consider $\rho_R : V \rightarrow V \otimes A$, $\rho_R(v) = R^1 v R^2 \otimes R^3 = v_{[0]} \otimes v_{[1]}$. Then $(V, \rho_R) \in \mathcal{YD}^{A^e}$, and the associated solution of the quantum Yang-Baxter equation is*

$$\Omega_R = \Omega_{(V, \rho_R)} : V \otimes V \rightarrow V \otimes V, \quad \Omega_R(v \otimes w) = R^1 w R^2 \otimes R^3 v.$$

Proof. We show that $(V, \rho_R) \in \mathcal{M}^{A \otimes A}$, that is, ρ satisfies (2-3). (2) follows from (20). (3) is equivalent to

$$(R^1 v R^2)_{[0]} \otimes (R^1 v R^2)_{[1]} \otimes R^3 = R^1 v R^2 \otimes 1 \otimes R^3$$

and to

$$(22) \quad r^1 R^1 v R^2 r^2 \otimes r^3 \otimes R^3 = R^1 v R^2 \otimes 1 \otimes R^3.$$

Using (19-21), we obtain:

$$R^1 \otimes R^2 \otimes 1 \otimes R^3 \stackrel{(21)}{=} r^1 R^1 \otimes r^2 \otimes r^3 R^2 \otimes R^3 \stackrel{(19)}{=} r^1 R^1 \otimes R^2 r^2 \otimes r^3 \otimes R^3$$

and (22) follows. It follows from Proposition 2.2 that $(V, \rho) \in \mathcal{YD}^{A^e}$, and we are done if we can show that the left A -action on V given by (7) coincides with the original left A -action. This can be shown easily:

$$v_{[0]} a v_{[1]} = R^1 v R^2 a R^3 = a R^1 v R^2 R^3 = a v.$$

We used (19-20), combined with the fact that R is invariant under cyclic permutation of the tensor factors. \square

Several examples of R -matrices are presented in [1]. In particular, if A is an Azumaya algebra, then we have a unique R -matrix. Applying Proposition 4.3 to [1, Example 2.8], we obtain the following.

Example 4.4. Let $A = M_n(k)$ be a matrix algebra and V an $M_n(k)$ -bimodule. Then the map

$$\Omega : V \otimes V \rightarrow V \otimes V, \quad \Omega(v \otimes w) = \sum_{i,j,k=1}^n e_{ij} w e_{ki} \otimes e_{jk} v$$

is a solution of the quantum Yang-Baxter equation. e_{ij} is the elementary matrix with 1 in the (i, j) -position and 0 elsewhere.

Problem 4.5. Let V be a finite dimensional vector space over a field k and let $\Omega \in \text{End}(V^{(2)})$ be a solution of the quantum Yang-Baxter equation such that $\Omega^3 = \Omega$. Does there exist an algebra A and a right $A \otimes A$ -coaction on V such that Ω is given by (17)?

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REFERENCES

- [1] A. L. Agore, S. Caenepeel, G. Militaru, Braidings on the category of bimodules, Azumaya algebras and epimorphisms of rings, arXiv:1108.2575.
- [2] G. Böhm, Hopf algebroids, *Handbook Algebra* **6** (2009), 173–235.
- [3] G. Böhm, S. Caenepeel, K. Janssen, Weak bialgebras and monoidal categories, *Comm. Algebra*, to appear.
- [4] G. Böhm, F. Nill, K. Szlachányi, Weak Hopf algebras I. Integral theory and C^* -structure, *J. Algebra* **221** (1999), 385–438.
- [5] T. Brzeziński, The structure of corings. Induction functors, Maschke-type theorem, and Frobenius and Galois properties, *Algebr. Representat. Theory* **5** (2002), 389–410.
- [6] T. Brzeziński, Flat connections and (co)modules, in “New Techniques in Hopf Algebras and Graded Ring Theory”, S. Caenepeel and F. Van Oystaeyen (eds), Universa Press, Wetteren, 2007, pp. 35–52.
- [7] T. Brzeziński, A note on flat noncommutative connections, preprint 2011.
- [8] T. Brzeziński and R. Wisbauer, “Corings and comodules”, *London Math. Soc. Lect. Note Ser.* **309**, Cambridge University Press, Cambridge, 2003.
- [9] S. Caenepeel, Galois corings from the descent theory point of view, *Fields Inst. Comm.* **43** (2004), 163–186.
- [10] M. Cipolla, Discesa fedelmente piatta dei moduli, *Rend. Circ. Mat. Palermo (2)* **25** (1976), 43–46.
- [11] A. Grothendieck, Technique de Descente I, *Sém. Bourbaki*, exp. **190** (1959-1960).
- [12] A. Joyal, R. Street, Braided tensor categories, *Adv. Math.* **102** (1993), 20–78.
- [13] A. Joyal, R. Street, Tortile Yang-Baxter operators in tensor categories, *J. Pure Appl. Algebra* **71** (1991), 43–51.
- [14] C. Kassel, “Quantum Groups”, *Graduate Texts Math.* **155**, Springer Verlag, Berlin, 1995.
- [15] M. A. Knus and M. Ojanguren, “Théorie de la descente et algèbres d’Azumaya”, *Lecture Notes in Math.* **389**, Springer Verlag, Berlin, 1974.
- [16] S. Majid, Quantum double for quasi-Hopf algebras, *Lett. Math. Phys.* **45** (1998), 1–9.
- [17] S. Majid, Representations, duals and quantum doubles of monoidal categories, *Rend. Circ. Math. Palermo (2) Suppl.* **26** (1991), 197–206.
- [18] S. Montgomery, “Hopf algebras and their actions on rings”, American Mathematical Society, Providence, 1993.
- [19] F. Nill, Axioms for weak bialgebras, preprint q-alg/9805104.
- [20] P. Schauenburg, Duals and doubles of quantum groupoids (\times_R -Hopf algebras), *Contemp. Math.* **267** (2000), 273–299.
- [21] P. Schauenburg, The monoidal center construction and bimodules, *J. Pure Applied Algebra*, **158** (2001), 325–346.
- [22] K. Szlachányi, Finite quantum groupoids and inclusions of finite type, *Fields Inst. Comm.* **30** (2001), 393–407.

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